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2001 J. Phys. A: Math. Gen. 34 10293

(http://iopscience.iop.org/0305-4470/34/47/327)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10293–10306 PII: S0305-4470(01)27628-8

Dynamical localization criterion for driven two-level systems

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Received 25 July 2001, in final form 12 September 2001
Published 16 November 2001
Online at stacks.iop.org/JPhysA/34/10293

Abstract
In this paper we consider a two-level system under the effect of an external time-dependent field. We give a precise criterion for dynamical localization. We then apply our result to the cases of external *ac–dc* and bichromatic field.

PACS numbers: 31.15.Md, 02.30.Gp, 02.30.Hq

1. Introduction

The study of the dynamics of two-level systems subject to external time-dependent fields has attracted an increasing interest in these last years and it remains an active field in the present days (see, for instance [BJL, BO, BW, DL, GH, SZH, WC, WFZ, WWM, WZ] and the references therein). The main address concerns the understanding of the effect of tunnelling destruction for critical values of the external field parameters.

Driven two-level systems have been the subject of great interest since the pioneering study by Rabi who solved the problem of a two-level spin system in a circularly polarized magnetic field. In the case of linearly polarized fields the Hamiltonian takes the form as in equation (1) [DL, GH].

Two-level systems have also been introduced in order to describe, for instance, the dynamics of a quantum state for Schrödinger equation with double-well potential under the effect of an external driven field. If the external frequencies are not too large, as we assume to be in this paper, then the ionization effect could be neglected and we can reduce our analysis to a bi-dimensional space, that is the two-level system of the type given below. This is not the case of a high-frequency regime where a coupling term between the two fundamental states with the continuous spectrum must be taken into account (see, for instance [BMRZ, GS1, GS2]).

The equation that describes the dynamics of a two-level driven system can be written as

$$i\hbar\dot{\phi} = H_1\phi \qquad H_1 = \epsilon\sigma_1 + f(t)\sigma_3 \tag{1}$$

where ϵ is the splitting parameter, $\dot{\phi}$ denotes the derivative of ϕ with respect to the time t,

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}$$

$f(t)$ is the driving force that depends on time and $\sigma_{1,3}$ are the two Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hereafter, we set the units such that $\hbar = 1$.

One of the most interesting effects associated with equation (1) is the so-called dynamical localization effect that we are going to briefly explain here. When the external field is absent then the solution of the unperturbed equation

$$\mathrm{i}\dot{\phi} = \epsilon\sigma_1\phi \qquad \phi(0) = \phi^\circ$$

for an initially localized wavefunction, e.g. $|\phi_1^\circ| = 1$ and $\phi_2^\circ = 0$, is a periodic function with period $t_B = \frac{2\pi}{\epsilon}$ and $|\phi_1(t)|$ and $|\phi_2(t)|$ periodically assumes the values 0 and 1. It has been found [GDJH] that in a periodically driven two-level system such an initially localized wavefunction has, for certain values of the system's parameters, a nearly localized behaviour for times of the order of the unperturbed period t_B, as if it was 'frozen' by the periodic perturbation. This phenomenon is usually called the dynamical localization effect.

Recently, several results concerning the dynamical localization effect, both numerical and analytical, have been given in the case of ac–dc external field [WZ] and [WFZ], bichromatic external field [WR] and random external field [SZH]. At present the full description of the occurrence of the dynamical localization effect is not yet given in a general setting and the only rigorous result is given in the case of a monochromatic external field [GDJH].

In this paper, by means of quite simple techniques, we compute (theorem 1) the solution of equation (1) by means of a convergent power series for times of the order of the period $t_B = 2\pi/\epsilon$ and we also give a general criterion (see equation (8) after corollary 2 and remark 4) for the occurrence of the dynamical localization effect in the limit of small splitting and for the fixed-driven field strength. Our result is very general and it could be applied to any driven field.

In fact in section 4 we consider the case of an ac–dc external field. In such a model we have, in agreement with the results obtained by [WZ] and [WFZ] by means of other techniques, dynamical localization for any external frequency ω such that $2a_0/\omega$ is not an integer number (a_0 here denotes the strength of the dc field); for resonant frequencies such that $2a_0/\omega = n$ is an integer number we then have a periodic behavior of the solution $\phi(t)$ with fundamental period $2\pi/J_n(2a/\omega)$ where a denotes the strength of the ac field and J_n is the nth Bessel function of first type.

In section 5 we consider the case of a bichromatic external field. In such a model we prove that the effect of dynamical localization is strongly sensitive with respect to the ratio between the two driving frequencies (see theorem 9 and remark 12). This fact could suggest the appearance of quantum chaos as discussed in [EF, PDG, WR].

2. Main results

For our purposes it is convenient to write the original equation (1) in a different form by means of the transformation

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\phi \qquad \text{where} \quad \alpha(t) = \int_0^t f(\xi)\,\mathrm{d}\xi.$$

Then equation (1) takes the form

$$\mathrm{i}\dot{\psi} = H_2\psi \qquad H_2 = \epsilon \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\sigma_1 \mathrm{e}^{-\mathrm{i}\alpha\sigma_3} = \begin{pmatrix} 0 & \epsilon \mathrm{e}^{2\mathrm{i}\alpha} \\ \epsilon \mathrm{e}^{-2\mathrm{i}\alpha} & 0 \end{pmatrix} \tag{2}$$

that is

$$\begin{aligned} \mathrm{i}\dot{\psi}_1 &= \epsilon \mathrm{e}^{2\mathrm{i}\alpha(t)}\psi_2 \\ \mathrm{i}\dot{\psi}_2 &= \epsilon \mathrm{e}^{-2\mathrm{i}\alpha(t)}\psi_1 \end{aligned} \tag{3}$$

with initial condition $\psi(0) = \psi^\circ = \phi^\circ$.

We are going to compute the solution $\psi(t)$ of equation (2) for any t in the interval $[0, t_B]$, where $t_B = 2\pi/\epsilon$, in the limit of small ϵ.

If $\psi_1^\circ = \psi_2^\circ = 0$ then the solutions of equation (3) are simply given by $\psi_1(t) \equiv \psi_2(t) \equiv 0$.

In the following, let $\psi_1^\circ \neq 0$ or $\psi_2^\circ \neq 0$, in particular we can assume, without loss of generality, that $f(t)$ is a real-valued piece-wise continuous function and that

$$\psi_1^\circ \neq 0 \qquad \text{and} \qquad |\psi_2^\circ| \leqslant |\psi_1^\circ|.$$

We have that:

Theorem 1. *If the limit*

$$\lim_{t\to+\infty} \frac{1}{t} I(t) = \hat{I} \qquad \textit{where} \quad I(t) = \int_0^t \mathrm{e}^{2\mathrm{i}\alpha(\xi)}\,\mathrm{d}\xi \tag{4}$$

exists, then for any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the solution $\psi_1(t)$ has the form

$$\psi_1(t) = \psi_1^\circ \exp\left[\int_0^t v(\xi)\mathrm{e}^{2\mathrm{i}\alpha(\xi)}\,\mathrm{d}\xi\right]$$

where $v(t)$ is given by the uniformly convergent series for any $t \in [0, \tau^*/\epsilon]$:

$$v(t) = \sum_{n=1}^{\infty} v_n(t)\epsilon^n \tag{5}$$

for any $\tau^* < \tau_{\max}$ fixed and where $\tau_{\max} = \max[1/(52\sqrt{|\hat{I}|}), 1/2]$ if $\hat{I} \neq 0$, $\tau_{\max} = +\infty$ if $\hat{I} = 0$, and

$$\begin{aligned} v_1(t) &= c_1 = -\mathrm{i}\frac{\psi_2^\circ}{\psi_1^\circ} \\ v_2(t) &= -\int_0^t \left[c_1^2 \mathrm{e}^{2\mathrm{i}\alpha(\xi)} + \mathrm{e}^{-2\mathrm{i}\alpha(\xi)}\right]\mathrm{d}\xi \\ v_n(t) &= -\sum_{\ell=1}^{n-1}\int_0^t \mathrm{e}^{2\mathrm{i}\alpha(\xi)} v_\ell(\xi)v_{n-\ell}(\xi)\,\mathrm{d}\xi \qquad n \geqslant 3. \end{aligned} \tag{6}$$

In particular, we have that:

Corollary 2. *If $\hat{I} = 0$ then for any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ then*

$$|\psi_1(t)|^2 - |\psi_1^\circ|^2 \leqslant \eta \qquad \textit{and} \qquad |\psi_2(t)|^2 - |\psi_2^\circ|^2 \leqslant \eta \quad \forall\, t \in [0, t_B]. \tag{7}$$

Remark 3. We underline that if the limit (4) exists then $\mathrm{e}^{2\mathrm{i}\alpha(t)}$ is usually called a KBM-vector field with average $\hat{I}$ (see [SV] ch 3). In particular, in the case $\hat{I} = 0$ then (7) could be seen as a result of general averaging techniques.

Remark 4. As a result of corollary 2 it follows that the condition

$$\hat{I} = 0 \tag{8}$$

implies dynamical localization: if $|\phi_1^\circ| = 1$ and $\phi_2^\circ = 0$ then $\phi_1(t)| \sim 1$ and $|\phi_2(t)| \sim 0$, for any $t \in [0, t_B]$, as ϵ goes to zero. That is the initial state is 'frozen' by the external field.

Remark 5. By making use of the same techniques we can extend our analysis to the case of ϵ depending on t. Indeed, now let the splitting parameter be given by

$$\epsilon = \epsilon(t) = \hat{\epsilon} h(t)$$

where $\hat{\epsilon}$ is a real constant parameter and $h(t)$ is a given bounded function. Then, in the limit of $\hat{\epsilon}$ that goes to zero the localization criterion holds where we define now

$$I(t) = \int_0^t h(\xi) \mathrm{e}^{2\mathrm{i}\alpha(\xi)} \, \mathrm{d}\xi.$$

In particular, if $f(t) = c$ is a constant function and $h(t)$ is a periodic function with period T, then we have dynamical localization for any non-resonant c such that $cT/2\pi$ is not an integer number.

3. Proofs of theorem 1 and corollary 2

In order to compute the solution of equation (3) we observe that (3) is equivalent to the second-order differential equation

$$\ddot{\psi}_1 - 2\mathrm{i} f(t) \dot{\psi}_1 + \epsilon^2 \psi_1 = 0 \tag{9}$$

with initial condition

$$\psi_1(0) = \psi_1^\circ \qquad \text{and} \qquad \dot{\psi}_1(0) = -\mathrm{i}\epsilon\psi_2^\circ.$$

Now, if we set

$$\psi_1(t) = \psi_1^\circ \exp\left[\int_0^t v(\xi) \mathrm{e}^{2\mathrm{i}\alpha(\xi)} \, \mathrm{d}\xi\right] \qquad \alpha(t) = \int_0^t f(\xi) \, \mathrm{d}\xi$$

we have

$$\psi_2(t) = \frac{\mathrm{i}}{\epsilon} v(t) \psi_1(t)$$

and that equation (9) takes the form of the generalized Riccati equation

$$\dot{v} + \mathrm{e}^{2\mathrm{i}\alpha} v^2 + \mathrm{e}^{-2\mathrm{i}\alpha} \epsilon^2 = 0 \qquad v(0) = -\mathrm{i}\epsilon \frac{\psi_2^\circ}{\psi_1^\circ}. \tag{10}$$

We have that

Lemma 6. *The formal solution of equation (10) is given by means of the formal series (5) where the function $v_n(t)$ are defined in (6).*

Proof. The proof of this lemma is quite standard, see for instance [B]. By substituting $v(t)$ for (5) in equation (10) we obtain the following sequence of equations

$$\begin{aligned} &\dot{v}_1 = 0 \qquad \dot{v}_2 + \mathrm{e}^{2\mathrm{i}\alpha} v_1^2 + \mathrm{e}^{-2\mathrm{i}\alpha} = 0 \\ &\dot{v}_n + \mathrm{e}^{2\mathrm{i}\alpha} \sum_{\ell=1}^{n-1} v_\ell v_{n-\ell} = 0 \qquad n \geqslant 3 \end{aligned} \tag{11}$$

which have solutions

$$
\begin{aligned}
v_1(t) &= c_1 \qquad v_2(t) = -\int_0^t \left[c_1^2 e^{2i\alpha(\xi)} + e^{-2i\alpha(\xi)}\right] d\xi + c_2 \\
v_n(t) &= -\sum_{\ell=1}^{n-1} \int_0^t e^{2i\alpha(\xi)} v_\ell(\xi) v_{n-\ell}(\xi)\, d\xi + c_n \quad n \geqslant 3.
\end{aligned}
\tag{12}
$$

In particular, since $v(0) = -i\epsilon \frac{\psi_2^\circ}{\psi_1^\circ}$ for any ϵ, we set

$$
c_1 = -i\frac{\psi_2^\circ}{\psi_1^\circ} \qquad \text{and} \qquad c_n = 0 \quad \forall\, n \geqslant 2
$$

to obtain (6). □

Now, we are going to prove the convergence of the series (5) for times of the order of the unperturbed period $t_B = 2\pi/\epsilon$. To this end, we prove that:

Lemma 7. *Let $v_n(t)$ given as in lemma 5, for any $\alpha \in [0, 1]$ and any $C > 0$ fixed, let $t_1 = C\epsilon^{-\alpha}$, then*

$$
|v_n(t)| \leqslant (1 + |c_1|)C_1^{n-1}\epsilon^{-\alpha(n-1)} \qquad \forall\, n \geqslant 1 \quad \textit{for } t \in [0, t_1] \tag{13}
$$

and

$$
|v_n(t)| \leqslant C_2^n t^{(n-1)/2}[I]_t^{(n-1)/2} \qquad \forall\, n \geqslant 1 \quad \textit{for } t \geqslant t_1 \tag{14}
$$

where

$$
C_1 = (1 + |c_1|)C \qquad \textit{and} \qquad C_2 = (1 + |c_1|)\left[\frac{41}{18} + \frac{1}{6}\sqrt{19643}\right] < 52
$$

$c_1 = -i\psi_2^\circ/\psi_1^\circ$ *is such that* $|c_1| \leqslant 1$, and $[I]_t$ *is a monotone non-decreasing function defined as*

$$
[I]_t = \begin{cases} t & \textit{if} \quad t < t_1 \\ \max\left[t_1, \max_{\xi\in[t_1,t]} |I(\xi)|\right] & \textit{if} \quad t \geqslant t_1 \end{cases} \tag{15}
$$

and

$$
I(t) = \int_0^t e^{2i\alpha(\xi)}\, d\xi. \tag{16}
$$

Proof. We prove this lemma by induction. Since $v_1(t) = c_1$, then (13) is true for $n = 1$ and for any t. For $n = 2$, from (6) it follows that

$$
\begin{aligned}
|v_2(t)| &= \left|\int_0^t \left[c_1^2 e^{2i\alpha(\xi)} + e^{-2i\alpha(\xi)}\right] d\xi\right| \\
&\leqslant \int_0^t \left|c_1^2 e^{2i\alpha(\xi)}\right| d\xi + \int_0^t \left|e^{-2i\alpha(\xi)}\right| d\xi \leqslant (1 + |c_1|^2)t \\
&\leqslant (1 + |c_1|)^2 t.
\end{aligned}
$$

Let us assume that for some $n \geqslant 2$

$$
|v_\ell(t)| \leqslant (1 + |c_1|)^\ell t^{(\ell-1)} \qquad \forall\, t \geqslant 0 \tag{17}
$$

is true for any $\ell \leqslant n$, then from (6) it follows that

$$
\begin{aligned}
|v_{n+1}(t)| &\leqslant \sum_{\ell=1}^{n} \int_0^t |v_\ell(\xi)||v_{n+1-\ell}(\xi)|\, d\xi \\
&\leqslant \sum_{\ell=1}^{n} (1 + |c_1|)^{n+1} \int_0^t \xi^{n-1}\, d\xi = (1 + |c_1|)^{n+1} t^n.
\end{aligned}
$$

Therefore, estimate (17) is true for $n+1$, too. In particular, for $t \in [0, t_1]$ the estimate (13) follows for $C_1 = (1+|c_1|)C$. In order to prove (14) let

$$u_n(t) = n^2 v_n(t) \qquad n \geqslant 1.$$

Then equations (11) take the form

$$\dot{u}_1 = 0 \qquad \dot{u}_2 + 4\left[\mathrm{e}^{2\mathrm{i}\alpha}u_1^2 + \mathrm{e}^{-2\mathrm{i}\alpha}\right] = 0$$
$$\dot{u}_n + \mathrm{e}^{2\mathrm{i}\alpha}\sum_{\ell=1}^{n-1}\frac{n^2}{\ell^2(n-\ell)^2}u_\ell u_{n-\ell} = 0 \qquad n \geqslant 3$$

which have solutions

$$u_1(t) = c_1 = -\mathrm{i}\frac{\psi_2^\circ}{\psi_1^\circ}$$
$$u_2(t) = -4\int_0^t \left[c_1^2\mathrm{e}^{2\mathrm{i}\alpha(\xi)} + \mathrm{e}^{-2\mathrm{i}\alpha(\xi)}\right]\mathrm{d}\xi$$
$$u_n(t) = -\sum_{\ell=1}^{n-1}\frac{n^2}{\ell^2(n-\ell)^2}U_{n-1,\ell}(t) \qquad n \geqslant 3$$

where

$$U_{n,\ell}(t) = \int_0^t \mathrm{e}^{2\mathrm{i}\alpha(\xi)}u_\ell(\xi)u_{n+1-\ell}(\xi)\,\mathrm{d}\xi.$$

Now, we are going to prove that

$$|u_n(t)| \leqslant (4\tilde{C})^{n-1}t^{(n-1)/2}[I]_t^{(n-1)/2}(1+|c_1|)^n \tag{18}$$

for any $\tilde{C} \geqslant \left[\frac{41}{72} + \frac{1}{24}\sqrt{19643}\right]$ and any $t \geqslant t_1$, from which immediately follows the estimate (14). To this end we check that (18) is true for $n = 1$ and $n = 2$; indeed:

$$|u_1(t)| = |c_1| \leqslant 1 + |c_1|$$

and

$$|u_2(t)| = 4\left|c_1^2 I(t) + \bar{I}(t)\right| \leqslant 4|I(t)|\left(1+|c_1|^2\right) \leqslant 4\tilde{C}t^{1/2}[I]_t^{1/2}(1+|c_1|)^2$$

since $\tilde{C} > 1$ and, by definition,

$$|I(t)| \leqslant [I]_t \quad \forall\, t \geqslant t_1 \qquad \text{and} \quad |I(t)| \leqslant t. \tag{19}$$

Let us assume that for some $n \geqslant 2$ the estimate (18) is true for any $\ell \leqslant n$, then we prove that (18) is true for $n+1$, too. Indeed, by integration by parts, we have that:

$$u_{n+1}(t) = -\sum_{\ell=1}^{n}\frac{(n+1)^2}{\ell^2(n+1-\ell)^2}U_{n,\ell}^1(t) - \sum_{\ell=1}^{n}\frac{(n+1)^2}{\ell^2(n+1-\ell)^2}U_{n,\ell}^2(t)$$

where

$$U_{n,\ell}^1(t) = I(t)u_\ell(t)u_{n+1-\ell}(t)$$

and

$$U_{n,\ell}^2(t) = -2\int_0^t I(\xi)\dot{u}_\ell(\xi)u_{n+1-\ell}(\xi)\,\mathrm{d}\xi.$$

From the above equations it follows that

$$
\begin{aligned}
&U^2_{n,1}(t) = 0\\
&U^2_{n,2}(t) = 2\int_0^t I(\xi)4\left[e^{2i\alpha(\xi)}u_1^2(\xi) + e^{-2i\alpha(\xi)}\right]u_{n-1}(\xi)\,d\xi\\
&U^2_{n,\ell}(t) = 2\sum_{j=1}^{\ell-1}\frac{\ell^2}{j^2(\ell-j)^2}\int_0^t e^{2i\alpha(\xi)}I(\xi)u_j(\xi)u_{\ell-j}(\xi)u_{n+1-\ell}(\xi)\,d\xi \qquad \ell\geqslant 3.
\end{aligned}
$$

By induction we have that

$$
\begin{aligned}
|U^1_{n,\ell}(t)| &\leqslant |I(t)|(4\tilde{C})^{n-1}t^{(n-1)/2}[I]_t^{(n-1)/2}(1+|c_1|)^{n+1}\\
&\leqslant \frac{1}{4\tilde{C}}\left[(4\tilde{C})^n t^{n/2}[I]_t^{n/2}(1+|c_1|)^{n+1}\right]
\end{aligned}
$$

since (19). Moreover,

$$
\begin{aligned}
|U^2_{n,2}(t)| &\leqslant 8(1+|c_1|^2)(1+|c_1|)^{n-1}(4\tilde{C})^{n-2}[I]_t^{(n-2)/2}\int_0^t |I(\xi)|\xi^{(n-2)/2}\,d\xi\\
&\leqslant \frac{2}{(n-2)\tilde{C}^2}\left[(4\tilde{C})^n t^{n/2}[I]_t^{n/2}(1+|c_1|)^{n+1}\right]
\end{aligned}
$$

since, for $t_1 \leqslant t$, we have that

$$
\begin{aligned}
\int_0^t |I(\xi)|\xi^{(n-2)/2}\,d\xi &\leqslant \int_0^{t_1} |I(\xi)|\xi^{(n-2)/2}\,d\xi + \int_{t_1}^t |I(\xi)|\xi^{(n-2)/2}\,d\xi\\
&\leqslant \frac{2}{n}t_1^{(n+2)/2} + [I]_t\frac{2}{n-2}t^{n/2}\\
&\leqslant \frac{2}{n-2}t^{n/2}\left[\frac{n-2}{n}t_1 + [I]_t\right]\\
&\leqslant \frac{4}{n-2}t^{n/2}[I]_t.
\end{aligned}
$$

Finally, we have that for $\ell \geqslant 3$

$$
\begin{aligned}
\left|U^2_{n,\ell}(t)\right| &\leqslant 2(4\tilde{C})^{n-2}(1+|c_1|)^{n+1}\sum_{j=1}^{\ell-1}\frac{\ell^2}{j^2(\ell-j)^2}\int_0^t |I(\xi)|\xi^{(n-2)/2}[I]_\xi^{(n-2)/2}\,d\xi\\
&\leqslant \frac{(4\tilde{C})^n}{8\tilde{C}^2}(1+|c_1|)^{n+1}\sum_{j=1}^{\ell-1}\frac{\ell^2}{j^2(\ell-j)^2}[I]_t^{(n-2)/2}\int_0^t |I(\xi)|\xi^{(n-2)/2}\,d\xi\\
&\leqslant \frac{(4\tilde{C})^n}{2\tilde{C}^2(n-2)}(1+|c_1|)^{n+1}t^{n/2}[I]_t^{n/2}F(\ell-1)
\end{aligned}
$$

where we denote

$$
F(n) = \sum_{\ell=1}^{n}\frac{(n+1)^2}{\ell^2(n+1-\ell)^2}.
$$

Therefore:

$$
|u_{n+1}(t)| \leqslant \mathbf{F}_n(4\tilde{C})^n t^{n/2}[I]_t^{n/2}(1+|c_1|)^{n+1} \qquad n\geqslant 3
$$

where

$$
\mathbf{F}_n = \left[\frac{F(n)}{4\tilde{C}} + \frac{2}{(n-2)\tilde{C}^2} + \frac{1}{2(n-2)\tilde{C}^2}\sum_{\ell=3}^{n}\frac{(n+1)^2}{\ell^2(n+1-\ell)^2}F(\ell-1)\right].
$$

If we observe that

$$F = \max_n F(n) = F(3) = \frac{41}{9}$$

then

$$\mathbf{F}_n \leqslant \frac{F}{4\tilde{C}} + \frac{2}{(n-2)\tilde{C}^2} + \frac{F^2}{2(n-2)\tilde{C}^2} \leqslant \frac{F}{4\tilde{C}} + \frac{2}{\tilde{C}^2} + \frac{F^2}{2\tilde{C}^2} \leqslant 1$$

for any $n \geqslant 3$ if

$$\tilde{C} \geqslant \frac{F}{8} + \frac{1}{8}\sqrt{33F^2 + 128} = \left[\frac{41}{72} + \frac{1}{24}\sqrt{19643}\right].$$

□

The function $[I]_t$ defined in equation (15) has the following property:

Lemma 8. *Let α, C and $t_1 = C\epsilon^{-\alpha}$ be as in lemma 6; if the limit*

$$\lim_{t\to+\infty} \frac{1}{t} I(t) = \hat{I} \tag{20}$$

exists, then for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ we have that

$$[I]_t \leqslant \begin{cases} t & \text{if} \quad t < t_1 \\ t_1 & \text{if} \quad t_1 \leqslant t \leqslant \max[t_1, t_1/(|\hat{I}| + \delta)]. \\ (|\hat{I}| + \delta)t & \text{if} \quad t_1/(|\hat{I}| + \delta) \leqslant t \end{cases} \tag{21}$$

Proof. From (20) we have that for any fixed $\delta > 0$ there exists $\bar{t}$ such that $|I(t)| \leqslant (|\hat{I}| + \delta)t$ for any $t \geqslant \bar{t}$. Now, let ϵ_0 be such that

$$\bar{t} < C\epsilon_0^{-\alpha} \leqslant C\epsilon^{-\alpha} = t_1 \quad 0 < \epsilon < \epsilon_0.$$

From this and from the definition of $[I]_t$ the lemma follows. □

Now, the proof of theorem 1 is an immediate consequence of lemmas 6 and 7.

In order to prove the corollary 2, let

$$\psi_{1,n}(t) = \psi_1^\circ \exp\left[\sum_{m=1}^n \epsilon^m \int_0^t \mathrm{e}^{2\mathrm{i}\alpha(\xi)} v_m(\xi)\,\mathrm{d}\xi\right]. \tag{22}$$

Then, we have that

$$|\psi_{1,n}(t)| \leqslant |\psi_1^\circ| \left[\sum_{m=1}^n \epsilon^m V_m(t)\right] \qquad V_m(t) = \left|\int_0^t \mathrm{e}^{2\mathrm{i}\alpha(\xi)} v_m(\xi)\,\mathrm{d}\xi\right|.$$

Now, $V_1(t) \leqslant |I(t)||c_1|$ for any t. Moreover, for $m \geqslant 2$ we have that if $t \leqslant t_1$ then

$$V_m(t) \leqslant \int_0^t |v_m(\xi)|\,\mathrm{d}\xi \leqslant (1 + |c_1|)C_1^{m-1}\epsilon^{-\alpha(m-1)}C\epsilon^{-\alpha} \leqslant C_1^m \epsilon^{-m\alpha}$$

and, if $t_1 \leqslant t$ then

$$\begin{aligned} V_m(t) \leqslant \int_0^{t_1} |v_m(\xi)|\,\mathrm{d}\xi + \int_{t_1}^t |v_m(\xi)|\,\mathrm{d}\xi \leqslant (1 + |c_1|)C_1^{m-1}\epsilon^{-\alpha(m-1)}C\epsilon^{-\alpha} \\ + \int_{t_1}^t C_2^m \xi^{(m-1)/2}[I]_\xi^{(m-1)/2}\,\mathrm{d}\xi \leqslant C_1^m \epsilon^{-m\alpha} + \tilde{V}_m(t) \end{aligned}$$

$$\tilde{V}_m(t) = C_2^m t^{(m+1)/2}[I]_t^{(m-1)/2}$$

where, from the above lemma, we have that

$$\tilde{V}_m(t) \leqslant \begin{cases} C_2^m t_1^m \delta^{-(m+1)/2} & \text{if} \quad t_1 \leqslant t < t_1/\delta \\ C_2^m t^m \delta^{(m-1)/2} & \text{if} \quad t_1/\delta \leqslant t \end{cases}.$$

Hence,

$$V_m(t) \leqslant \begin{cases} C_4^m \epsilon^{-m\alpha} & \text{if} \quad t < t_1 \\ C_5^m \delta^{-(m+1)/2} \epsilon^{-m\alpha} & \text{if} \quad t_1 \leqslant t < t_1/\delta \\ C_6^m \epsilon^{-m\alpha} + C_7^m \delta^{(m-1)/2} \epsilon^{-m} & \text{if} \quad t_1/\delta \leqslant t \leqslant t_B = \frac{2\pi}{\epsilon} \end{cases}$$

for some constant C_4, C_5, C_6 and C_7. Therefore:

$$V(t) = \sum_{m=1}^{\infty} \epsilon^m V_m(t)$$

converges for ϵ and δ small enough and such that $\epsilon^{1-\alpha} \ll \delta$. In particular, we have that

$$V(t) \leqslant \epsilon t_1 + \sum_{m=2}^{\infty} C_4^m \epsilon^{m(1-\alpha)} \leqslant \tilde{C}_4 \epsilon^{1-\alpha} \qquad \text{if} \quad t < t_1$$

$$\begin{aligned} V(t) &\leqslant \epsilon t_1 + \sum_{m=2}^{\infty} C_5^m \delta^{-(m+1)/2} \epsilon^{m(1-\alpha)} \\ &\leqslant C\epsilon^{1-\alpha} + \tilde{C}_5 \epsilon^{2(1-\alpha)} \delta^{-3/2} \qquad \text{if} \quad t_1 \leqslant t < t_1/\delta \end{aligned}$$

$$\begin{aligned} V(t) &\leqslant \epsilon \delta t_B + \sum_{m=2}^{\infty} \left[C_6^m \epsilon^{m(1-\alpha)} + C_7^m \delta^{(m-1)/2} \right] \\ &\leqslant 2\pi\delta + \tilde{C}_6 \epsilon^{2(1-\alpha)} + \tilde{C}_7 \delta^{1/2} \qquad \text{if} \quad t_1/\delta \leqslant t \leqslant t_B \end{aligned}$$

for some positive constants $\tilde{C}_4$, $\tilde{C}_5$, $\tilde{C}_6$ and $\tilde{C}_7$. Hence,

$$V(t) \leqslant C_8 \delta^{1/2} \qquad \forall t \in [0, t_B]$$

for some constant C_8 and

$$|\psi_1(t) - \psi_1^\circ| \leqslant |\psi_1^\circ| e^{V(t)} V(t) \leqslant |\psi_1^\circ| \tilde{C}_8 \delta^{1/2} \qquad \forall t \in [0, t_B]$$

for some constant $\tilde{C}_8$. The corollary follows from this fact and from the fact that the quantity $E(t) = |\psi_1(t)|^2 + |\psi_2(t)|^2$ is an integral of motion; i.e. $E(t) \equiv E(0)$. Indeed:

$$\begin{aligned} \dot{E} &= \dot{\bar{\psi}}_1 \psi_1 + \bar{\psi}_1 \dot{\psi}_1 + \dot{\bar{\psi}}_2 \psi_2 + \bar{\psi}_2 \dot{\psi}_2 \\ &= i\epsilon e^{-2i\alpha} \bar{\psi}_2 \psi_1 - i\epsilon e^{2i\alpha} \bar{\psi}_1 \psi_2 + i\epsilon e^{2i\alpha} \bar{\psi}_1 \psi_2 - i\epsilon e^{-2i\alpha} \bar{\psi}_2 \psi_1 = 0 \end{aligned}$$

where $\bar{\psi}_1$ (respectively $\bar{\psi}_2$) denotes the complex conjugation of ψ_1 (respectively ψ_2).

4. Dynamical localization for driving ac–dc fields

Here, we consider an external field of the form

$$f(t) = a_0 + a\cos(\omega t) \tag{23}$$

where a_0 is the strength of the dc field and where a and ω are, respectively, the amplitude and the frequency of the ac field. We consider, at first, the case $a_0 = 0$.

4.1. Case $a_0 = 0$

This model has been introduced by [GDJH]. We have that $\alpha(t) = a/\omega \sin(\omega t)$ is a periodic function. Let

$$\hat{I} = \lim_{t\to+\infty} \frac{1}{t} I(t).$$

Then, we have that (see formula (9.1.18) in [AS])

$$\hat{I} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} e^{2a\mathrm{i}\,\sin(\omega\xi)/\omega}\, d\xi = J_0(2a/\omega)$$

where J_0 is the zeroth Bessel function. As a result of corollary 2 and remark 4 we have dynamical localization when a and ω are such $J_0(2a/\omega) = 0$.

We underline that in such a model an expression for the approximate solution of equation (3) could be also obtained by means of the average perturbative method [GDJH]. In such a case we have that α (t) is a periodic function with period $T = 2\pi/\omega$ and we can approximate the solution ψ of (3) by means of the solution $\hat{\psi}$ of the 'average' system given by

$$\begin{cases} \mathrm{i}\dot{\hat{\psi}}_1 & = & \Delta\hat{\psi}_2 \\ \mathrm{i}\dot{\hat{\psi}}_2 & = & \bar{\Delta}\hat{\psi}_1 \end{cases} \qquad \hat{\psi}(0) = \psi^\circ \tag{24}$$

where $\Delta = \epsilon J_0(2a/\omega)$. It is easy to check that equation (24) has solutions

$$\hat{\psi}_1(t) = \cos(\Delta t)\psi_1^\circ - \mathrm{i}\sin(\Delta t)\psi_2^\circ \qquad \hat{\psi}_2(t) = -\mathrm{i}\sin(\Delta t)\psi_1^\circ + \cos(\Delta t)\psi_2^\circ$$

By means of the average theorem [A2], we have that, in the limit of small ϵ, there exists a constant c such that

$$|\hat{\psi}_{1,2}(t) - \psi_{1,2}(t)| \leqslant c\epsilon \qquad \forall t \in [0, t_B].$$

Hence, the solution $\psi(t)$ of the perturbed equation (3) exhibits a periodic behaviour with period $t_B^{per} = 2\pi/\Delta$; in particular, when the eternal frequency assumes a resonant value such that $J_0(2a/\omega) = 0$ then the periodic motion disappears and we have dynamical localization; i.e.:

$$|\psi_{1,2}^\circ - \psi_{1,2}(t)| \leqslant c\epsilon \qquad \forall t \in [0, t_B].$$

4.2. Case $a_0 \neq 0$

In such a case we have that $\alpha(t) = a_0 t + a/\omega \sin(\omega t)$. This problem has been discussed in [WZ, WFZ] where the numerical and analytical evidence of the occurrence of the dynamical localization effect has been given; here, making use of the criterion given in corollary 2 and remark 4, we give the rigorous proof of the occurrence of the dynamical localization effect when $2a_0/\omega$ is not an integer number. Let

$$\begin{aligned} \hat{I} = \hat{I}_{a_0,a} &= \lim_{t\to+\infty} \frac{1}{t} \int_0^t \exp\{2\mathrm{i}[a_0\xi + a/\omega \sin(\omega\xi)]\}\, d\xi \\ &= \lim_{t'\to+\infty} \frac{1}{t} \int_0^{t} \exp\{2\mathrm{i}[a_0\xi + a\sin(\xi)]/\omega\}\, d\xi. \end{aligned}$$

Then:

Theorem 9. *We have that*

$$\hat{I}_{a_0,a} = J_n(-2a/\omega) \qquad \text{if} \quad 2a_0/\omega = n \in Z \tag{25}$$

where J_n is the nth Bessel function, and

$$\hat{I}_{a_0,a} = 0 \qquad \text{if} \quad 2a_0/\omega \notin Z. \tag{26}$$

Proof. If $2a_0/\omega = n$ is an integer number then $e^{2i\alpha(t)}$ is a periodic function with period 2π. In such a case we have that $\hat{I}_{a_0,a} = J_n(-2a/\omega)$ where J_n is the *nth* Bessel function. This result is a consequence of the formula (see section 12.3 in [AS])

$$\frac{1}{\pi}\int_0^\pi e^{\pm i(\nu x-\beta\sin x)}\,dx = \mathbf{J}_\nu(\beta) \pm i\mathbf{E}_\nu(\beta)$$

where $\mathbf{E}_\nu$ are the Weber functions and $\mathbf{J}_\nu$ are the Anger functions such that $\mathbf{J}_n = J_n$ when n is integer, and from the fact that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2i\alpha(\xi)}\,d\xi &= \frac{1}{2\pi}\left[\int_0^\pi e^{2i\alpha(\xi)}\,d\xi + \int_{-\pi}^0 e^{2i\alpha(\xi)}\,d\xi\right]\\ &= \frac{1}{2\pi}\left[\int_0^\pi e^{2i\alpha(\xi)}\,d\xi + \int_0^\pi e^{-2i\alpha(\xi)}\,d\xi\right] = J_n(-2a/\omega).\end{aligned}$$

If $2a_0/\omega \in R - Q$ then $\hat{I}_{a_0,a} = 0$. Indeed, let

$$g(x,y) = e^{ix}e^{2ia/\omega\sin(y)}$$

from which

$$\hat{I}_{a_0,a} = \lim_{t\to\infty}\frac{1}{t}\int_0^t g(\omega_1 t, \omega_2 t)\,dt$$

where $\omega_1 = 2a_0/\omega$ and $\omega_2 = 1$. Since $\omega_1/\omega_2 \notin Q$ then $\hat{I}_{a_0,a}$ coincides with spatial average value [A1] of g on the torus T^2, hence

$$\hat{I}_{a_0,a} = \frac{1}{|T^2|}\int_{T^2} g(x,y)\,dx\,dy = 0.$$

Finally, if $2a_0/\omega \in Q - Z$ let $2a_0/\omega = n/m$ where n and m are two integer numbers, which have no common divisor, such that $m \neq 1$, then $\hat{I}_{a_0,a} = 0$ for any a. Indeed, $2i\alpha(t)$ is a periodic function with period $2m\pi$ and

$$\begin{aligned}\hat{I}_{a_0,a} &= \lim_{t\to\infty}\frac{1}{t}\int_0^t \exp[ni\xi/m + 2ia/\omega\sin(\xi)]\,d\xi\\ &= \frac{1}{2m\pi}\int_{-m\pi}^{m\pi}\exp[ni\xi/m + 2ia/\omega\sin(\xi)]\,d\xi\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[ni\xi + 2ia/\omega\sin(m\xi)]\,d\xi.\end{aligned}$$

Now, from formulas (9.1.42) and (9.1.43) in [AS], we have that

$$\hat{I}_{a_0,a} = \frac{1}{\pi}\left[\sum_{k=1}^\infty a_k J_{2k}(2a/\omega) + i\sum_{k=0}^\infty b_k J_{2k+1}(2a/\omega)\right]$$

where all the coefficients a_k and b_k are zero since $n \neq km$:

$$a_k = \int_{-\pi}^{\pi} e^{in\varphi}\cos(2mk\varphi)\,d\varphi = 0$$

and

$$b_k = \int_{-\pi}^{\pi} e^{in\varphi}\sin[(2k+1)m\varphi]\,d\varphi = 0.$$

The proof of the theorem is completely given. □

As a result of corollary 2 and of theorem 8 it follows that dynamical localization occurs for any value of the external field parameters with the exception of the case $2a_0/\omega = n$ integer.

For such values we expect to observe a quasi-periodic motion with fundamental period given by $2\pi/J_n(-2a/\omega)$. The physical reason for this result is clear enough for a double-well model. Indeed, the dc field produces a localization result [CL] and the splitting between the two new fundamental states proportionally increases with respect to the dc field strength. The ac field does not affect this localization effect provided that the external frequency does not satisfy a resonant condition, that is the external frequency is not an integer multiply of the splitting between the two new fundamental states. For resonant values of the external frequency, the ac field couples the two new fundamental states and the beating effect between the two wells is restored (but with a different period).

5. Dynamical localization for driving bichromatic field

Here, we consider an external field given by two different monochromatic fields with same amplitude η and different frequencies ω_1 and ω_2:

$$f(t) = \frac{1}{2}\eta\,[\cos(\omega_1 t) - \cos(\omega_2 t)]\,.$$

Therefore,

$$\alpha(t) = \int_0^t f(\xi)\,\mathrm{d}\xi = -\frac{1}{2}\eta\left[\frac{\sin(\omega_1 t)}{\omega_1} - \frac{\sin(\omega_2 t)}{\omega_2}\right]$$

and

$$\hat{I} = \hat{I}_{\eta,\omega_1,\omega_2} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \exp\left\{-\mathrm{i}\eta\left[\frac{\sin(\omega_1\xi)}{\omega_1} - \frac{\sin(\omega_2\xi)}{\omega_2}\right]\right\}\mathrm{d}\xi.$$

If we set

$$\Lambda = \frac{\eta}{\omega_2} \qquad \text{and} \qquad \Omega = \frac{\omega_2}{\omega_1}$$

then it follows that

$$\hat{I} = \hat{I}_{\eta,\omega_1,\omega_2} = \hat{I}_{\Lambda,\Omega} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \exp\{-\mathrm{i}\Lambda\,[\Omega\sin q - \sin(\Omega q)]\}\,\mathrm{d}q.$$

We have that:

Theorem 10. *If $\Omega = 1$ then $\hat{I}_{\Lambda,1} = 1$ for any Λ. If $\Omega \neq 1$ then*

$$\hat{I}_{\Lambda,\Omega} = J_0(z_1)J_0(z_2) + R_{\Lambda,\Omega} \tag{27}$$

where $z_1 = \Lambda\Omega = \eta/\omega_1$, $z_2 = \Lambda = \eta/\omega_2$ and

$$R_{\Lambda,\Omega} = \begin{cases} 0 & \text{if} \quad \Omega \in R - Q \\ \sum_{\ell=-\infty:\ell\neq 0}^{\infty} J_{n\ell}(z_1)J_{m\ell}(z_2) & \text{if} \quad \Omega = \frac{n}{m} \in Q \end{cases} \tag{28}$$

where n and m are two integer numbers which have no common divisor.

Proof. In the case $\Omega = 1$ the result follows immediately. In the case $\Omega \neq 1$ we recall that (see formulas (9.1.42) and (9.1.43) in [AS])

$$\mathrm{e}^{\mathrm{i}z\sin\varphi} = \sum_{k=-\infty}^{\infty} J_k(z)\mathrm{e}^{\mathrm{i}k\varphi}.$$

Then, if we set

$$z_1 = \Lambda\Omega \qquad z_2 = \Lambda \qquad \varphi_1 = q \qquad \varphi_2 = \Omega q$$

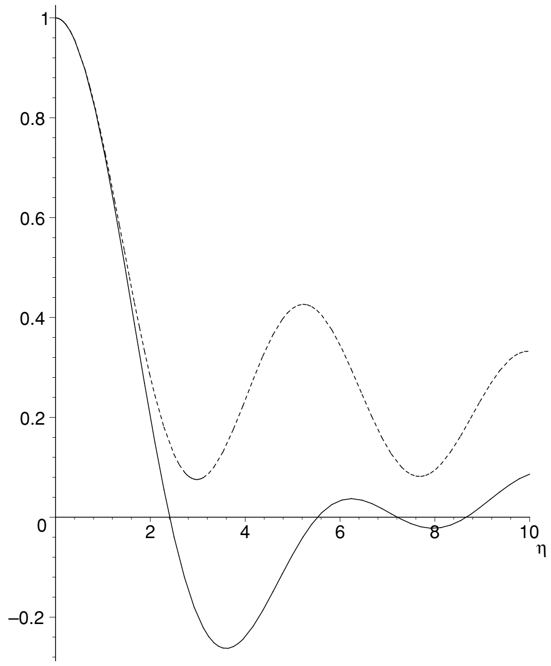

Figure 1. In this figure we plot the graphs of the function $\hat{I}_{\eta,\omega_1,\omega_2}$ versus η for $\frac{\omega_1}{\omega_2} = \frac{1}{3}$ (broken line) and $\frac{\omega_1}{\omega_2} = \frac{1.001}{3}$ (solid line). Dynamical localization effect, corresponding to the zeros of the function $\hat{I}$, that appears for some values of η in the case $\frac{\omega_1}{\omega_2} = \frac{1.001}{3}$ disappears drastically when ω_1 goes from 1.001 to 1.

it follows that

$$\hat{I}_{\Lambda,\Omega} = \sum_{k,\ell=-\infty}^{\infty} J_k(z_1) J_\ell(z_2) D_{k,\ell}$$

where

$$D_{k,\ell} = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{i(\ell\Omega - k)q}\, dq = \begin{cases} 0 & \text{if} \quad \Omega\ell \neq k \\ 1 & \text{if} \quad \Omega\ell = k \end{cases}.$$

From this fact and from the estimate

$$|J_n(z)| \leqslant |z/2|^n/n! \tag{29}$$

(formula (9.1.62) in [AS]) then (27) follows and the series giving $R_{\Lambda,\Omega}$ is absolutely convergent. □

Remark 11. From (28) and (29) it follows that

$$\hat{I}_{\eta,\omega_1,\omega_2} \approx J_0(\eta/\omega_1) J_0(\eta/\omega_2) \tag{30}$$

if ω_1 and ω_2 are incommensurate numbers (in such a case (30) is an exact formula) or if $\frac{\omega_1}{\omega_2} = \frac{n}{m} \in Q$ where n and m are two integer numbers large enough, which have no common divisor.

Remark 12. Let j_n be the *nth* zero of the Bessel function J_0; e.g.: $j_0 = 2.404825558$, $j_1 = 5.520078110$, $j_2 = 8.653727913, \ldots$. Then, if the external field is such that $\eta/\omega_1 = j_n$ or $\eta/\omega_2 = j_n$ and if ω_1 and ω_2 are such that (30) holds, we have dynamical localization.

Remark 13. In order to understand how the dynamical localization effect depends on the ratio ω_1/ω_2, we compare the functions $\hat{I}_{\eta,\omega_1,\omega_2}$, for $\eta \in [0, 10]$, where, respectively, $\frac{\omega_1}{\omega_2} = \frac{1}{3}$ and $\frac{\omega_1}{\omega_2} = \frac{1.001}{3}$. In the second case we can apply (30), while in the first case we have the remainder term $R_{\Lambda,\Omega}$ which gives a significant contribution. As appears in figure 1, only in the case of $\frac{\omega_1}{\omega_2} = \frac{1.001}{3}$ we have the occurrence of dynamical localization effect for some values of η.

Acknowledgments

This work is partially supported by the Italian MURST and INDAM-GNFM. I thank Professor Joao C A Barata and Professor Vincenzo Grecchi for helpful discussions and remarks. I also thank one of the referees for useful comments.

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